Theorem (1) > If T is a continuous linear transformation from a nls E into a nls F then $\ker(T)$ is a closed linear subspace of E.

Proof:- We have $\ker T = \{x \in E : T(x) = 0\}$. Let $x_1, x_2 \in \ker T$ and $\alpha_1, \alpha_2 \in K$, then

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$$
$$= \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0$$

$\therefore \alpha_1 x_1 + \alpha_2 x_2 \in \ker T$.

So $\ker T$ is a linear subspace of T.

Now, $\{0\}$ is closed in F. Since T is continuous $T^{-1}\{0\} = \ker T$ is a closed set in E.

Theorem (2):> A linear transformation T from a nls E to a nls F is continuous iff it is continuous at the origin.

Proof:- Suppose T is continuous. Then it is continuous at every point of E. Hence T is continuous at the origin.

Conversely, let T be continuous at the origin then $\because T(0) = 0$ given $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - 0\| < \delta \Rightarrow \|T(x) - T(0)\| < \epsilon$

i.e. $\|x\| < \delta \Rightarrow \|T(x)\| < \epsilon$.

Now, for all $x, y \in E$, $\|x - y\| < \delta \Rightarrow \|T(x-y)\| = \|T(x) - T(y)\| < \epsilon$.

$\therefore T$ is uniformly continuous.

Hence T is continuous.

(Def n):- Let E be a nls over field K of real or Complex nos. Then K it self is a nls over K w.r.t. the norm defined

$\|\alpha\| = |\alpha|$ for all $\alpha \in K$. A linear transformation, T from E into K is called a linear functional on E.

Corollary:- A linear function T on a nls E iff it is Continuous it is Continuous at the origin.

(Def n) Bounded linear transformations:- A linear transformation T from a nls E into a nls F is said to be bounded if there exist a +ve real number m such that $\|T(x)\| \leq m\|x\|$ for all $x \in E$.

A linear functional T is a nls E is said to be bounded if there exists $m > 0$ such that -

$|T(x)| \leq m \cdot \|x\|$ for all $x \in E$.

Theorem

(1) A linear transformation T from a nls E into a nls F is Continuous iff it is bounded

or, ANo- A linear transformation T from a normed linear space E into a normed linear space F is Continuous iff T is bounded in the sense that exists a +ive real number m such that $\|T(x)\| \leq m \|x\|$ for all $x \in E$.

Proof - Suppose that T is continuous. If possible let T be not bounded. Then, for every +ive real no. m, there exists a point $x \in E$ such that $\|T(x)\| > m\|x\|$. In particular, for every +ive integer n, there exists $x_n \in E$, such that

$\|T(x_n)\| > n\|x_n\|$. We define,

$$y_n = \frac{x_n}{n\|x_n\|} \quad \text{Then } \|y_n\| = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

$\therefore \{y_n\}$ converges to 0. Now

$$\|T(y_n)\| = \left\|T\left(\frac{x_n}{n\|x_n\|}\right)\right\| = \frac{1}{n\|x_n\|} \|T(x_n)\| > 1. \text{ for}$$

all n.

$\therefore \{T(y_n)\}$ ~~does not~~ doesn't converge to $T(0) = 0$.

So T is not continuous at 0, This is contradiction. Hence T must be bounded.

Conversely, let T be bounded. Then there exists, $m > 0$ such that

$\|T(x)\| \leq m\|x\|$, for all $x \in E$.

Let $\epsilon > 0$ be given, we choose $\delta = \frac{\epsilon}{m}$

Then, $\|x\| < \delta \Rightarrow \|T(x)\| \leq m\delta = \epsilon$.

Therefore, T is continuous at orgin and So,

$\therefore T$ is a continuous.

(Defn) Bounded Set in a metric space:- Let (E, d) be a a metric space A subset A of E is said to be bounded if there exists $a \in E$ and $K > 0$ such that $d(x, a) \leq K$ for all $x \in A$.

i.e. $x \in S_K[a]$ for $x \in A$. i.e. $A \subseteq S_K[a]$. So, A is bounded iff it is contained in some closed sphere.

If E is a nls, then a subset A of E is bounded if there exists $a \in E$ & $K > 0$ such that

$$\|x - a\| \leq K \text{ for all } x \in A.$$

i.e., if $x \in S_K[a]$ for all $x \in A$ or if $A \subseteq S_K[a]$.

Theorem

QN. → A linear transformation T from a nls E into a nls F is continuous iff the image T(S) of the closed unit ball (unit sphere) $S = \{x \in E : \|x\| \leq 1\}$ of E is a bounded in F.

Proof :- Let T be continuous. Then it is bounded. So there exists $m > 0$ such that $\|T(x)\| \leq m\|x\|$ for all $x \in E$. Now, $x \in S \Rightarrow \|x\| \leq 1$.

$\therefore \|T(x)\| \leq m$ i.e. $T(x) \in S_m[0]$

$\therefore T(S) \subseteq S_m[0]$. Hence T(S) is a bounded set in F.

Conversely, Let T(S) be bounded in F. Then there exists a closed ball $S_\epsilon[0]$ such that $T(S) \subseteq S_\epsilon[0]$.

Now, if $x = 0$ then clearly $\|T(x)\| \leq \epsilon\|x\|$.

If $x \neq 0$, we choose $y = \dfrac{x}{\|x\|}$ then $\|y\| = 1$ and so $y \in S$. Hence $\|T(y)\| \leq \epsilon$

$$\left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq \epsilon$$

$$\text{or, } \frac{1}{\|x\|}\|T(x)\| \leq \epsilon$$

$$\text{or, } \|T(x)\| \leq \epsilon\|x\|$$

$\therefore T$ is bounded and so it is continuous